On the Brittleness of Evolutionary Algorithms

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Motivation

Evolutionary algorithms are algorithms.

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1. correctness
2. (expected) run time

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Evolutionary algorithms are function optimizers
(among other things). to Ken: Sorry!

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Analysis of Evolutionary Algorithms

1. convergence to global optimum √
2. expected optimization time

What Evolutionary Algorithm Should Be Analyzed?

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Example infinite population models to Adam: Sorry!

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Example population size 1

The (1+1) EA

for maximization of $f\colon \{0,1\}^n \to \mathbb{R}$

1. Initialization

   $t := 0$; Choose $x_t \in \{0,1\}^n$ uniformly at random.
2. Mutation

   $y := x_t$; Independently for each bit in y, flip this bit with probability $p_m = 1/n$.
3. Selection

   If $f(y) \geq f(x_t)$ Then $x_{t+1} := y$ Else $x_{t+1} := x_t$.
4. "Stopping Criterion"

   $t := t + 1$; Continue at line 2.

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Counter-Examples

- (1, 1) EA since selection independent of fitness
- "(1+1) EA" with 1-bit-mutation since no global search

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Observation toy problems (artificial examples) make sense

- for development of methods and analytical tools
- for illustration purposes
- as counter-examples
- as examples
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Answers Darrell's open question Can we learn from toy problems?
YES!

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Very well-known example $\text{ONEMAX}(x) = \sum_{i=1}^{n} x[i]$

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$$:\Leftrightarrow \exists w_0, w_1, \dots, w_n \in \mathbb{R}\colon \forall x \in \{0,1\}^n\colon f(x) = w_0 + \sum_{i=1}^{n} w_i \cdot x[i]$$

(w. l. o. g. $w_0 = 0,\ w_1 \geq w_2 \geq \cdots \geq w_n$)

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$:\Leftrightarrow \forall x \in \{0,1\}^n\colon f(x)$ max. $\vee\ \exists y\colon \mathrm{H}(x,y) = 1 \wedge f(y) > f(x)$

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Theorem $\exists$unimodal f: $\mathrm{E}\left(T_{(1+1)\text{ EA},f}\right) = \Theta\left(n^{3/2} \cdot 2^{\sqrt{n}}\right)$

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Important first step

Analysis of $\mathrm{E}\left(T_{(1+1)\ \mathrm{EA}}, \textsc{BinVal}\right)$

with $\textsc{BinVal}(x) := \sum_{i=1}^{n} 2^{n-i} \cdot x[i]$

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left-most mutated bit decides about acceptance

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Extreme Case

$0111 \cdots 11 \rightsquigarrow 1000 \cdots 00$

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Proof rather simple and very elegant

due to powerful general method drift analysis

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using mutation probability $p_m = 1/(2n)$

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Worst Case Assumption This is always the case.

The Model — A bit more formal...

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Definition optimization time = first hitting time of 1^n

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Optimization time of PO-EA is upper bound
for $\mathrm{E}\left(T_{(1+1)\ \mathrm{EA},f}\right)$ for linear f
as observed by Jun He and Xin Yao

Drift Analysis

Central Notion distance

for Z = set of all populations ($Z = \{0,1\}^n$ for (1+1) EA)

define distance $d\colon Z \to \mathbb{R}_0^+$
with $d(P) = 0 \Leftrightarrow P$ contains optimum

Observation optimization time $T := \min\{t \mid d(P_t) = 0\}$

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Drift Theorem (He/Yao (2001))
$\Delta > 0 \Rightarrow \mathrm{E}(T) \leq M/\Delta$

Proof of the Drift Theorem

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Proof

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$$M \geq \mathrm{E}\left(\sum_{t=1}^{T} D_t\right)$$

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Proof of the Drift Theorem (cont.)

$$M \geq \sum_{i=1}^{\infty} \sum_{t=i}^{\infty} \mathrm{Prob}\,(T = t) \cdot \mathrm{E}\,(D_i \mid T = t)$$

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Proof of the Drift Theorem (cont.)

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\begin{aligned}
M &\geq \sum_{i=1}^{\infty} \sum_{t=i}^{\infty} \operatorname{Prob}(T = t) \cdot \operatorname{E}(D_i \mid T = t) \\
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&= \sum_{i=1}^{\infty} \operatorname{Prob}(T \geq i) \sum_{t=i}^{\infty} \operatorname{Prob}(T = t \mid T \geq i) \cdot \operatorname{E}(D_i \mid T = t \wedge T \geq i)
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Proof of the Drift Theorem (cont.)

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\begin{aligned}
M &\geq \sum_{i=1}^{\infty} \sum_{t=i}^{\infty} \mathrm{Prob}\left(T = t\right) \cdot \mathrm{E}\left(D_i \mid T = t\right) \\
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&= \sum_{i=1}^{\infty} \operatorname{Prob}(T\geq i)\operatorname{E}(D_i \mid T\geq i) \geq \Delta\cdot\sum_{i=1}^{\infty} \operatorname{Prob}(T\geq i)
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$$

Proof of the Drift Theorem (cont.)

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$$

thus $\mathrm{E}\left(T\right) \leq \frac{M}{\Delta}$ □

Drift Theorem for Lower Bounds

We made use of

- $M = \max\{d(P) \mid P \in Z\}$
- $\Delta_l = \min\{\mathrm{E}\left(d(P_{t-1}) - d(P_t) \mid T \geq t\right)\}$

and had

$$
\begin{aligned}
M &\geq \sum_{i=1}^{\infty} \sum_{t=i}^{\infty} \mathrm{Prob}\left(T = t\right) \cdot \mathrm{E}\left(D_i \mid T = t\right) \\
&= \sum_{i=1}^{\infty} \mathrm{Prob}\left(T \geq i\right) \mathrm{E}\left(D_i \mid T \geq i\right) \\
&\geq \Delta_l \cdot \sum_{i=1}^{\infty} \mathrm{Prob}\left(T \geq i\right) = \Delta_l \cdot \mathrm{E}\left(T\right)
\end{aligned}
$$

Changes for a Lower Bound Method

Observation just two inequalities "in wrong direction"

1. $M \geq \sum \cdots$ mit $M = \max d(P) \mid P \in Z$
2. $\sum \cdots \geq \Delta_l \cdot \sum \cdots$ mit
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obviously for lower bound

- $\Delta_u = \max\{\mathrm{E}\left(d(P_{t-1}) - d(P_t) \mid T \geq t\right)\}$
  sufficient for “$\leq$”
- instead of M we need lower bound M_u on $d(P_0)$

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  sufficient for "$\leq$"
- instead of M we need lower bound M_u on $d(P_0)$

this yields

- $\mathrm{E}\left(T \mid d(P_0) \geq M_u\right) \geq M_u/\Delta_u$
- $\mathrm{E}\left(T\right) \geq \mathrm{Prob}\left(d(P_0) \geq M_u\right) \cdot \mathrm{E}\left(T \mid d(P_0) \geq M_u\right) \geq$
  $\mathrm{Prob}\left(d(P_0) \geq M_u\right) \cdot M_u/\Delta_u$
- $\mathrm{E}\left(T\right) \geq \sum \mathrm{Prob}\left(d(P_0) \geq d\right) \cdot d/\Delta_u \geq \mathrm{E}\left(d(P_0)\right)/\Delta_u$

Results (of different kinds)

when using drift analysis for the optimization time of PO-EA

What distance function do we use?

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Theorem

$$\mathrm{E}\left(d(x_{t+1}) - d(x_t) \mid \text{OneMax}(x_t) = n - z\right) = \Theta\left(\left(\frac{z}{n}\right)^2\right)$$

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Sketch of Proof

$$\mathrm{E}\left(d(x_{t+1}) - d(x_t) \mid \textsc{OneMax}(x_t) = n - z\right)$$
$$= \left(\sum_{b_0=1}^{z} b_0 \binom{z}{b_0} \left(\frac{1}{n}\right)^{b_0} \left(1 - \frac{1}{n}\right)^{n-b_0}\right)$$
$$+ \sum_{b_0=1}^{z} \sum_{b_1=b_0+1}^{n-z} (b_0 - b_1) \binom{z}{b_0} \cdot \binom{n-z}{b_1} \cdot \left(\frac{1}{n}\right)^{b_0+b_1} \cdot \left(1 - \frac{1}{n}\right)^{n-b_0-b_1}$$

Sketch of Upper Bound

$$\begin{aligned}
& \mathrm{E}\left(d(x_{t+1}) - d(x_t) \mid \textsc{OneMax}(x_t) = n - z\right) \\
\leq \quad & \left(\sum_{b_0=1}^{z} b_0 \binom{z}{b_0} \left(\frac{1}{n}\right)^{b_0} \left(1 - \frac{1}{n}\right)^{n-b_0} \right) \\
& + \sum_{b_0=1}^{z} \sum_{b_1=0}^{n-z} (b_0 - b_1) \binom{z}{b_0} \binom{n-z}{b_1} \left(\frac{1}{n}\right)^{b_0+b_1} \left(1 - \frac{1}{n}\right)^{n-b_0-b_1}
\end{aligned}$$

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& \quad + \sum_{b_0=1}^{z} \sum_{b_1=0}^{n-z} (b_0 - b_1) \binom{z}{b_0} \binom{n-z}{b_1} \left(\frac{1}{n}\right)^{b_0+b_1} \left(1 - \frac{1}{n}\right)^{n-b_0-b_1} \\
=\ & \ldots \\
=\ & \frac{z}{n} - \frac{n-z}{n} + \left(1 - \frac{1}{n}\right)^{z} \frac{n-z}{n} \\
=\ & \frac{1}{n} \cdot \left(2z - n + \left(1 - \frac{1}{n}\right)^{z} (n - z)\right) \\
=\ & O\left(\left(\frac{z}{n}\right)^2\right) \quad \checkmark
\end{aligned}
$$

Sketch of Lower Bound

from proof of upper bound

$$\mathrm{E}\left(d(x_{t+1}) - d(x_t) \mid \textsc{OneMax}(x_t) = n - z\right)$$

$$= \frac{1}{n} \cdot \left(2z - n + \left(1 - \frac{1}{n}\right)^z (n - z)\right)$$

$$- \sum_{b_0=1}^{z} \sum_{b_1=0}^{b_0} (b_0 - b_1) \binom{z}{b_0} \binom{n-z}{b_1} \left(\frac{1}{n}\right)^{b_0+b_1} \left(1 - \frac{1}{n}\right)^{n-b_0-b_1}$$

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$$= \cdots = \Omega\left(\left(\frac{z}{n}\right)^2\right) \qquad \square$$

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&- \sum_{b_0=1}^{z} \sum_{b_1=0}^{b_0} (b_0 - b_1) \binom{z}{b_0} \binom{n-z}{b_1} \left(\frac{1}{n}\right)^{b_0+b_1} \left(1 - \frac{1}{n}\right)^{n-b_0-b_1} \\
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\end{aligned}$$

Drift Analysis leads to
- lower bound $\Omega(n)$:-(
- upper bound $O(n^3)$:-(

correct but not tight

Results for Simple d Reconsidered

With $d(x) := n - \text{ONEMAX}(x)$ we have

$$\mathrm{E}\left(d(x_{t+1}) - d(x_t) \mid \text{ONEMAX}(x_t) = n - z\right) = \Theta\left(\left(\frac{z}{n}\right)^2\right)$$

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more involved analysis following the lines of the $O(n \log n)$-proof in Droste/Jansen/Wegener (1998) leads to upper bound $O(n^2)$

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a slightly more involved conditional drift analysis
using drift for $\text{OneMax}(x_t) \geq n - \sqrt{n}$ and
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Observation He and Yao could not succeed in 2001
since this slight modification of the model
actually significantly increased the expected optimization time.

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Consider different distance measure

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$$\left. \cdot \mathbf{1}_{(b_1=0<b_0)\vee(b_1>b_0>0)} \cdot \left(\tilde{d}(z) - \tilde{d}(z + b_1 - b_0)\right) \right)$$

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upper bound $O\left(n^{3/2}\right)$

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Theorem $\quad \mathrm{E}\left(T_{\text{PO-EA}}\right) = \Theta\left(n^{3/2}\right)$

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simple, direct implementation, 100 runs, $n \in \{10, 20, \ldots, 2560\}$

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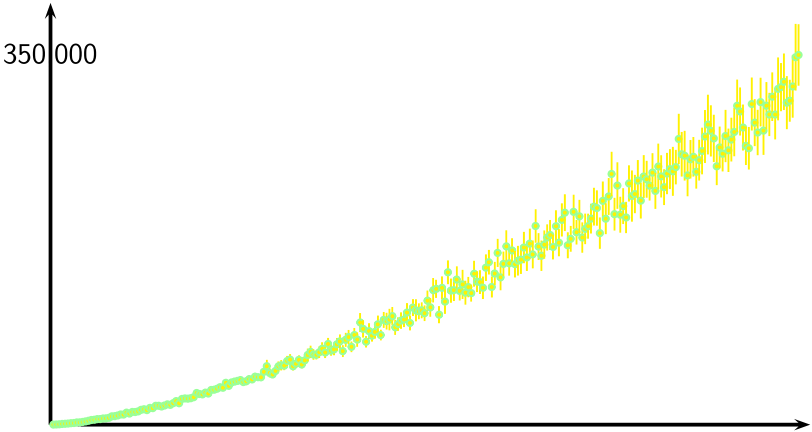
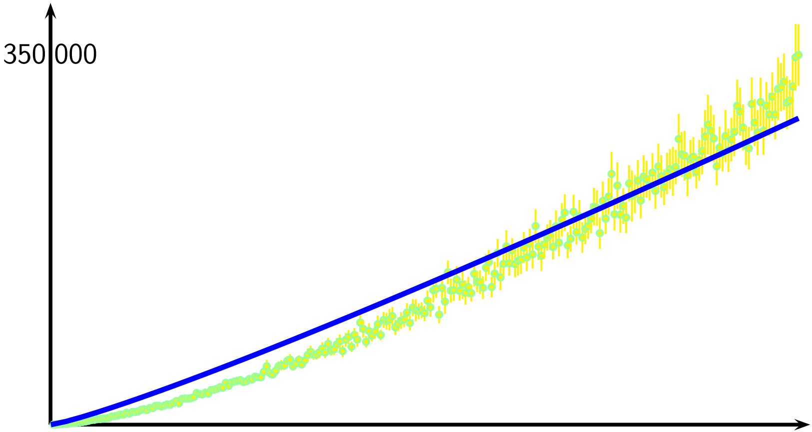


Empirical Results

simple, direct implementation, 100 runs, $n \in \{10, 20, \ldots, 2560\}$



with $14.346 \cdot n \cdot \ln(n)$

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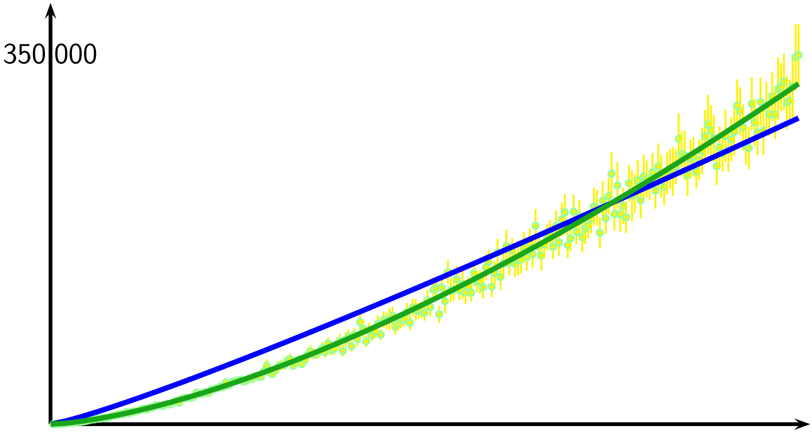

with $14.346 \cdot n \cdot \ln(n)$, $2.474 \cdot n^{3/2}$

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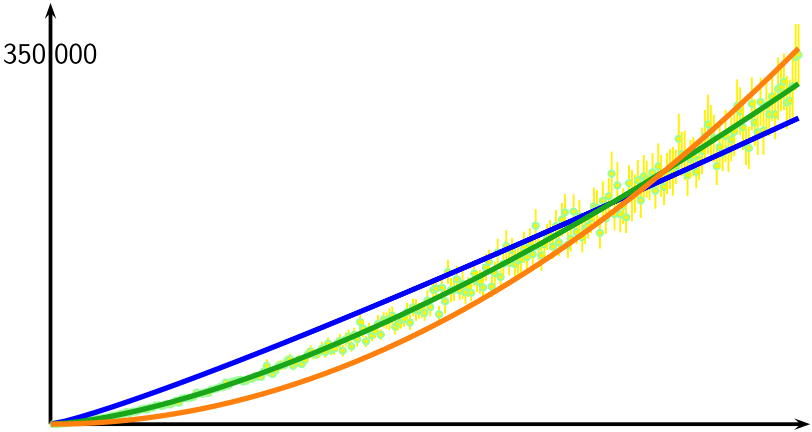

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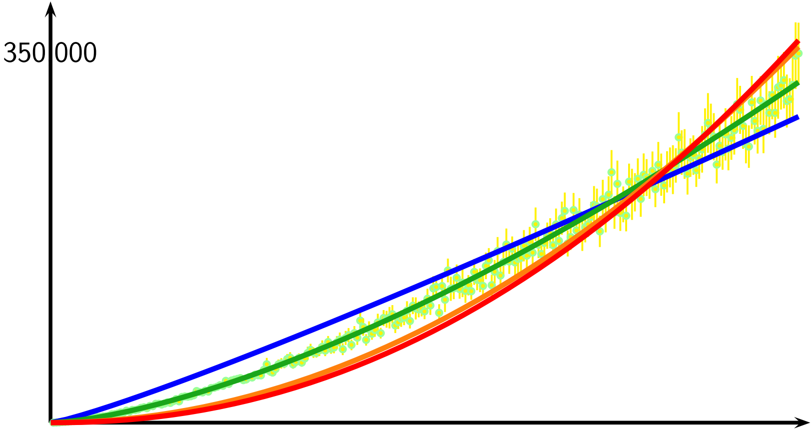

with $14.346 \cdot n \cdot \ln(n)$, $2.474 \cdot n^{3/2}$, $.054 \cdot n^2$, $.007 \cdot n^2 \cdot \ln(n)$

Summary

- still $\forall$linear f: $\mathrm{E}\left(T_{(1+1)\ \mathrm{EA},f}\right) = O(n \log n)$
- PO-EA abstracts from linear functions in a worst case way
- PO-EA is exact for $\textsc{BinVal}$ on $0111\cdots1$, mutations of single bits, pure mutations
- yet, there is no f such that PO-EA behaves exactly like (1+1) EA on f
- $\mathrm{E}\left(T_{(1+1)\ \mathrm{EA},f}\right) = \Theta(n^{3/2})$
- drift analysis is a powerful general tool

On the Choice of a Title

What has all this got to do with brittleness of evolutionary algorithms?

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PO-EA is an example where a quite small change of the mutation probability from $\frac{1}{n}$ to $\frac{1}{2n}$ changes the average optimization time considerably.

On the Brittleness of Evolutionary Algorithms

Thomas Jansen

Summary

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- $\mathrm{E}\left(T_{(1+1)\ \mathrm{EA},f}\right) = \Theta(n^{3/2})$
- drift analysis is a powerful general tool
- (1+1) EA extremely sensitive to changes of p_m