

Generic Finiteness of the Fixed Point Set for the Infinite Population Genetic Algorithm

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 - Definition of Transversality
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Representation of Populations

1 Simple Genetic Algorithm

2 Representation:

- Search Space: Ω and $|\Omega| = n$;
- Population Size: $r < \infty$
- Population Incidence Vector: $p = (p_1, p_2, \dots, p_n)^T$ where $p_i = \frac{v_i}{r}$ is the proportion of the i^{th} element in the population.
- Take r to ∞ to build dynamical systems model on simplex Λ .

The GA Map

- $G : \Lambda \rightarrow \Lambda$ is a composition

$$G(x) = U \circ C \circ F(x)$$

- F - selection operator
 - C - crossover
 - U - mutation
- $x \mapsto G(x)$ is a discrete time dynamical system.

Modelling Selection and Mutation

- Selection $F : \Lambda \rightarrow \Lambda$

Let f_i denote the fitness of $i \in \Omega$. Then $f = (f_1, f_2, \dots, f_n)$.

- Example:

$$F(x) := \frac{\text{diag}(f)x}{f^T x}.$$

- Assume F is C^1 .
- Mutation $U : \Lambda \rightarrow \Lambda$
 - U is an $n \times n$ matrix, where $U_{ij} > 0$ is the probability that j is mutated into i ($i, j \in \Omega$).

$$U(x) := Ux$$

Modelling Crossover

- Crossover $C : \Lambda \rightarrow \Lambda$

Let $C_k(x)$ be the probability that the individual $k \in \Omega$ results from applying crossover to the population x .

$$C_k(x) := x^T A_k x$$

- A_k symmetric $n \times n$ matrix.
- $(A_k)_{ij}$ is the probability that individual k will result from crossover applied to parents of type i and j .

$$C(x) = \begin{pmatrix} x^T A_1 x \\ x^T A_2 x \\ \vdots \\ x^T A_n x \end{pmatrix}$$

The Complete Operator

- $G : \Lambda \rightarrow \Lambda$
- Let $x \in \Lambda$, then $G(x) := C \circ U \circ F(x)$, where

$$F : \Lambda \rightarrow \Lambda;$$

$$U(x) = Ux;$$

$$C_k(x) = x^T A_k x.$$

•

$$G_k(x) = [F(x)]^T U^T A_k U F(x)$$

$$G(x) = \begin{pmatrix} (F(x))^T U^T A_1 U F(x) \\ (F(x))^T U^T A_2 U F(x) \\ \vdots \\ (F(x))^T U^T A_n U F(x) \end{pmatrix}$$

The Mixing Operator: $M = C \circ U$

- $M : \Lambda \rightarrow \Lambda$
- $M_k := U^T A_k U$.
- For $x \in \Lambda$,

$$G_k(x) = (F(x))^T U^T A_k U F(x) = (F(x))^T M_k F(x).$$

Definition

A quadratic map $M = (M_1, \dots, M_n)$ is called a **mixing operator** if

- for all i , M_i is positive, symmetric, and $\sum_i M_i = \mathbf{1}$.

The set of all mixing operators is denoted \mathcal{M} .

We extend all maps $H \in \{C, U, F, M, G\}$ to the positive cone in \mathbb{R}^n by $\mathbb{H}(u) := H\left(\frac{u}{\sum_i u_i}\right)$.

Definition (GA map)

$$G = M \circ F, \text{ for } M \in \mathcal{M}$$

is the complete operator for the genetic algorithm, or a GA map.

Definition

If $f(x) = x$, a point x is called a **fixed point** of f .

How does this relate to Genetic Algorithms?

- Action of the Genetic Algorithm
 $\iff \{x, G(x), G(G(x)), \dots\}$.
- GA converges to the same result independent of the initial seed \iff
 G has a globally attracting fixed point
- Global attractor for infinite population \Rightarrow Global attractor for finite population
- Convergence to a unique fixed point is automatic for quadratic maps that are contractions – our work considers GAs which are not contractions.

Genericity

Definition

*A property is **typical**, or **generic**, if it holds for an open and dense set of parameter values.*

An optimist's analogy. Good people are generic means:

- Open: For any good guy, there is a small circle of friends (all good) nearby.
- Dense: For any bad guy we happen to pick, for any sized neighborhood, there is a good guy nearby.

Theorem

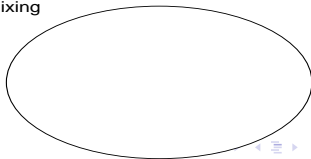
Let $G = M \circ F$ be a GA map. For a typical mixing operator $M \in \mathcal{M}$, G has finitely many fixed points.

Translation: For most mixing operators, G has finitely many fixed points. If, however, you happen to have picked a mixing operator for which this is not true, perturb it a little bit and then G WILL have finitely many fixed points.

Selection



Mixing

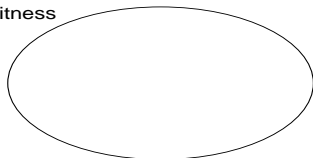


Theorem

(A. Wright, Vose 1995). For a generic fitness function, when selection is proportional, G has finitely many fixed points.

Translation: For most fitness functions, if G is defined using proportional selection, then G has finitely many fixed points. If, however, you happen to have picked a fitness function for which this is not true, perturb it a little bit and then G WILL have finitely many fixed points.

Fitness



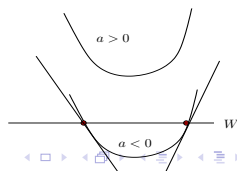
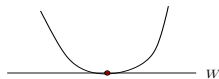
Mixing ●

Example

$$f(x, a) = x^2 + a, \quad a \in \mathbb{R}, \quad W = x\text{-axis in } \mathbb{R}^2$$

For $a \neq 0$, W intersects the graph of $f(x, a)$ either at two or no points – and this will remain true even under small perturbations of a . Term these a values “stable.”

The set of stable a values is open and dense in the set of all possible a values.



Example Revisited: Geometric Characterization

$$f(x, a) = x^2 + a, \quad a \in \mathbb{R}, \quad W = x\text{-axis in } \mathbb{R}^2$$

For the “stable” values of a :

- The sum of the tangent space to the graph of f and the tangent space to W at all points x of intersection generates the tangent space to \mathbb{R}^2 at x (\mathbb{R}^2).

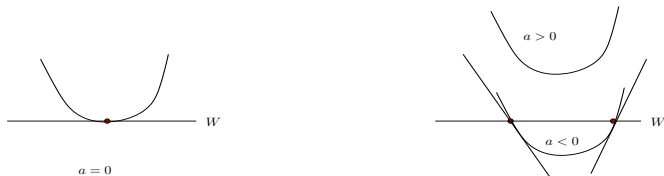
$$T_x f(\mathbb{R}^2) + T_x W = T_x(\mathbb{R}^2)$$

- That is, f is **transversal to W** .

Geometric Characterization Continued

- That the set of stable values of a values is open and dense in the set of all possible a values is a sophisticated way of saying:

f and W “cross” for the “stable” values of a and “touch” for the exceptional values of a .



Transversality

Definition

For smooth manifolds X, Y , let $f : X \rightarrow Y$ be C^1 with $W \subset Y$ a submanifold. We say f is **transversal to W at x** , denoted $f \pitchfork_x W$, if where $y = f(x)$ either $y \notin W$ or $y \in W$ and

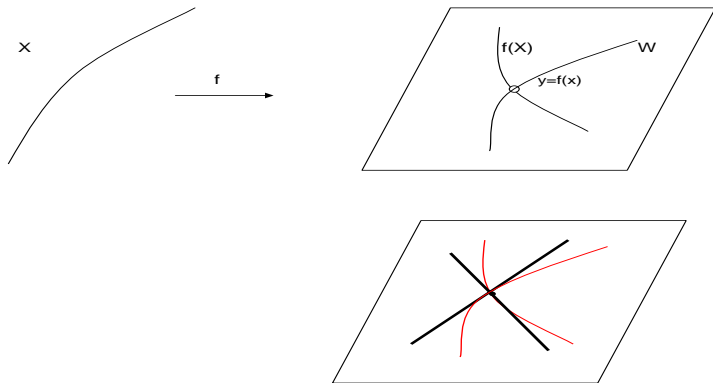
$$(T_x f)(T_x X) + T_y W = T_y Y.$$

Definition

We say f is **transversal to W** , denoted $f \pitchfork W$, if $f \pitchfork_x W$ for all $x \in X$.

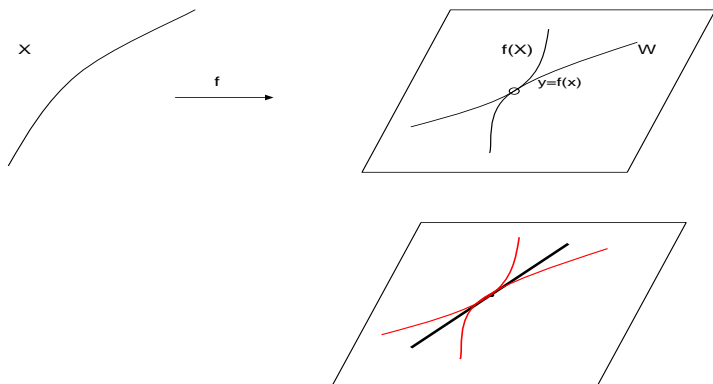
Definition of Transversality

Transverse



Definition of Transversality

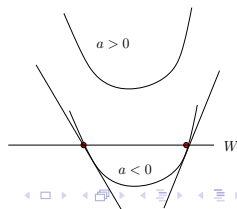
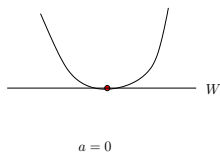
Not Transverse



Example: A family of maps

$$f_a : [-1, 1] \rightarrow \mathbb{R}^2 \text{ by } f_a(t) := (t, t^2 + a)$$

An exceptional value of a corresponds to non-transverse intersection of the graph of f and W (left figure); typical value of a corresponds to transverse intersection.



Transversal Density Theorem

Theorem

Let \mathcal{A}, X, Y be C^r manifolds and $W \subset Y$ a submanifold. Let $\rho_a : X \rightarrow Y$ be a family of maps such that the correspondence $ev(a, x) := \rho_a(x)$ is C^r . Define $\mathcal{A}_W \subset \mathcal{A}$ by

$$\mathcal{A}_W = \{a \in \mathcal{A} \mid \rho_a \pitchfork W\}.$$

Assume that

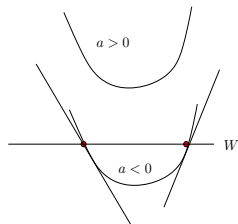
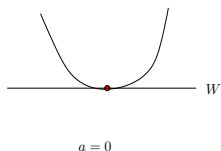
- 1 X has finite dimension m and W has finite codimension q in Y ;
- 2 $r > \max(0, m - q)$;
- 3 $ev(a, x) \pitchfork W$.

Then \mathcal{A}_W is residual (and hence dense) in \mathcal{A} .



Example for Transversal Density Theorem

In our example, the set $\mathcal{A}_W := \{a \neq 0\}$ and the set $\mathcal{A} = \mathbb{R}$. Density of \mathcal{A}_W in \mathcal{A} means that in an arbitrary neighborhood of the value $a = 0$, there is a value of the set \mathcal{A}_W .



Theorem which is really a Corollary to the Transversal Density Theorem

Theorem

Let X, Y be C^r manifolds ($r \geq 1$), $f : X \rightarrow Y$ a C^r map, $W \subset Y$ a C^r submanifold. Then if $f \pitchfork W$:

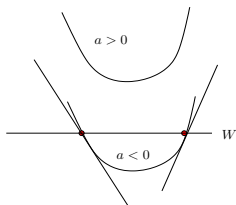
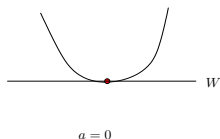
- 1 W and $f^{-1}(W)$ have the same codimension;
- 2 If W is closed and X is compact, $f^{-1}(\{W\})$ has only finitely many connected components.

Transversality and Density

Ex: W and $f^{-1}(W)$ have the same codimension

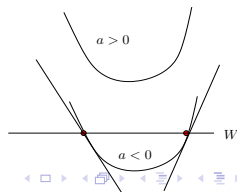
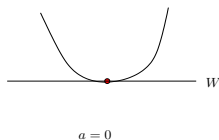
$$f_a : [-1, 1] \rightarrow \mathbb{R}^2 \text{ by } f_a(t) := (t, t^2 + a)$$

Let $W = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$, and $a \neq 0$.



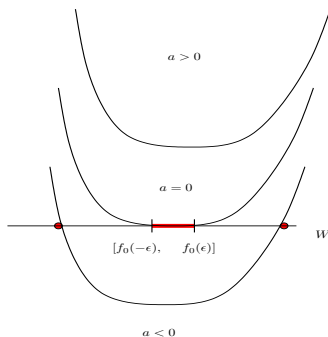
Example Continued

- The set $f_a^{-1}(W)$ consists of finitely many points. Two points when $a < 0$ and no points when $a > 0$. Each of these points is a zero-dimensional submanifold of the dimension 1 interval $[-1, 1]$.
- Thus, $f_a^{-1}(W)$ has codimension 1. Since W is a 1-dimensional submanifold of \mathbb{R}^2 , it also has codimension 1. Note that in our example, $f_0^{-1}(W) = 0$ and is a point.



Modifications of previous example.

The map f_0 maps a subinterval $(-\epsilon, \epsilon)$ into W .



Another Example

- Observe that the compactness of X is also necessary.
- Let $g : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $g(t) = (t, \sin t)$. Then $g^{-1}(0) = k\pi$ has infinitely many components in $X = \mathbb{R}$.

Openness of Transversal Intersection

Theorem

Let \mathcal{A}, X, Y be \mathcal{C}^1 manifolds with X finite dimensional, $W \subset Y$ a closed \mathcal{C}^1 submanifold, $K \subset X$ a compact subset of X , and $\rho_a : X \rightarrow Y$ be a family of maps such that the correspondence $ev(a, x) = \rho_a(x)$ is C^r . Then the subset $\mathcal{A}_{KW} \subset \mathcal{A}$ defined by $\mathcal{A}_{KW} = \{a \in \mathcal{A} \mid \rho_a \pitchfork_x W \text{ for } x \in K\}$ is open.

Example for Openness of Transversal Intersection

In our example, $K = X = [-1, 1]$ is compact and $\mathcal{A}_{KW} = \mathcal{A}_W := \{a \neq 0\}$. The set \mathcal{A}_{KW} is open in $\mathcal{A} = \mathbb{R}$ if a small enough neighborhood, $N_\epsilon(a)$, of the value $a \in \mathcal{A}_{KW}$, has $N_\epsilon(a) \subset \mathcal{A}$. That is, for any $a \neq 0$, there is a small interval $(a - \epsilon, a + \epsilon) \subset \mathcal{A}$.

Theorem

Theorem

Let $G = M \circ F$ be a GA map. For a typical mixing operator $M \in \mathcal{M}$, G has finitely many fixed points.

Set Up

- $\mathcal{A} = \mathcal{M}$, where \mathcal{M} is the set of mixing operators.
- $X = \Lambda \subset \mathbb{R}^n$ and $Y = \mathbb{R}_0^n := \{x \in \mathbb{R}^n \mid \sum_i x_i = 0\}$.
- Let $M \in \mathcal{M}$ and $x \in \Lambda$.

- Define $\rho_M(x) : \Lambda \rightarrow \mathbb{R}_0^n$, by

$$\rho_M(x) := (M \circ F - I)x.$$

- Define $\text{ev}_\rho(M, x) : \mathcal{M} \times \Lambda \rightarrow \mathbb{R}_0^n$ by

$$\text{ev}_\rho(M, x) := \rho_M(x) = M(F(x)) - x.$$

- $F, M, G, \rho_M, \text{ev}_\rho$ are each \mathcal{C}^1 .

Lemma

Lemma

For $\text{ev}_\rho(M, x) := M(F(x)) - x$, $\text{rank}(d(\text{ev}_\rho)) = n - 1$.

Shown by computation.

Lemma

Lemma

Let $\text{ev}_\rho : \mathcal{M} \times \Lambda \rightarrow \mathbb{R}_0^n$, $x \in \Lambda$, and $M \in \mathcal{M}$. Then $\text{ev}_\rho \pitchfork \{0\}$.

Show definition holds.

Proposition

Proposition

Let $\mathcal{M}_{\{0\}} := \{M \in \mathcal{M} \mid \rho_M \pitchfork \{0\}\}$. Then $\mathcal{M}_{\{0\}}$ is dense in \mathcal{M} . That is, the set of parameter values for which ρ_M is transversal to $\{0\}$ is dense in \mathcal{M} .

Use Transversal Density Theorem.

Proposition

Proposition

The set $\mathcal{M}_{\{0\}}$ is open in \mathcal{M} .

Use Openness of Transversal Intersection Theorem.

Proposition

Proposition

For generic $M \in \mathcal{M}$,

- ① $\rho_M \pitchfork \{0\}$. That is, the set of parameter values for which ρ_M is transversal to $\{0\}$ is open and dense in \mathcal{M} .
- ② The set of parameter values for which $\rho_M^{-1}(\{0\})$ has finitely many solutions is open and dense in \mathcal{M} .

For part (1) use previous two Lemmas. For part (2) use Corollary to the Transversal Density Theorem.

Proof of Main Result

Theorem

Let $G = M \circ F$ be a GA map. For a typical mixing operator $M \in \mathcal{M}$, G has finitely many fixed points.

Proof.

By previous Lemma, for $M \in \mathcal{M}_{\{0\}} \subset \mathcal{M}$, $\rho_M(x) = 0$ has finitely many solutions in Λ . That is, for generic $M \in \mathcal{M}$,

$$\rho_M(x) = M(F(x)) - x$$

has finitely many solutions in Λ . Thus solutions to $\rho_M(x) = 0$ correspond to fixed points of $G = M \circ F$. □

Summary of Result

- Given an arbitrary selection function and a typical mixing function, their composition has finitely many fixed points. (Infinite population model)
- Infinite population model vs Finite population models
 - correspondence not straightforward – likely depends on the details of the implementation
 - our result adds to the increasing body of evidence that the infinite population model can give qualitative insights into the functioning of the GA.

Genericity...

- Genericity of the finiteness of the fixed point set is expected for any reasonably rich model, consisting of iterations of a map on a compact space.
- For a given mixing operator, unless proven otherwise, it is reasonable to assume that G has finitely many fixed points.

Infinite vs Finite Population Models

- The perturbation from an infinite to a large finite population model can be viewed as a small perturbation of the infinite population model.
- For any finite population model which in the infinite population limit falls into the class we study, one can **reasonably assume** that each large population model in this class has finitely many fixed points.

Caveat behind “reasonably assume:”

...one can reasonably assume that each large population model in this class has finitely many fixed points.

- Generally more difficult to prove density results in a smaller parameter set as opposed to in a larger parameter set.
 - Fewer available perturbations to perturb off of “bad” parameter values (as is $a = 0$ in our example).
- Possibility exists that a specific class of mixing operators in which genetic algorithms with a finite number of fixed points will not be dense.



An additional contribution...

Differential Topology is useful in the study of Genetic Algorithms



Details for Proof of Main Result in Generic Finiteness of the Fixed Point Set for the Infinite Population Genetic Algorithm

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 - Set Up

- 2 Proof Outlines



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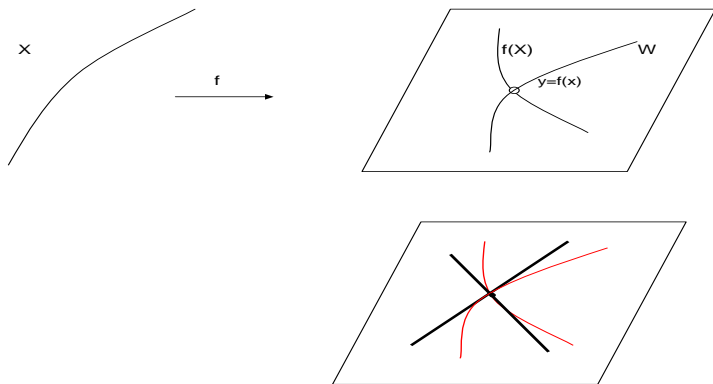
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Definition

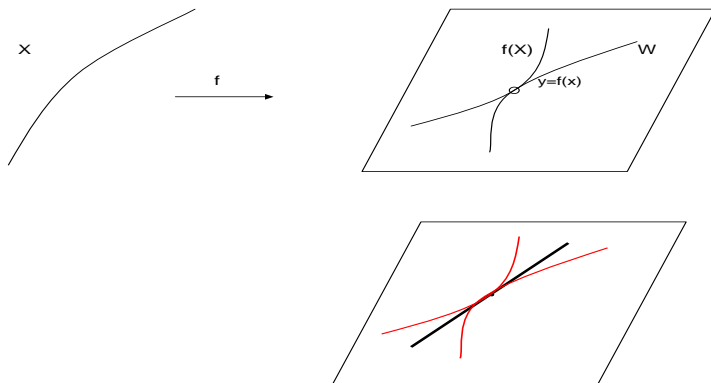
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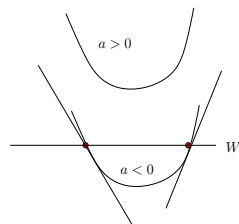
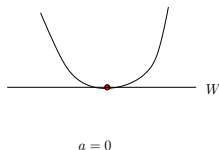
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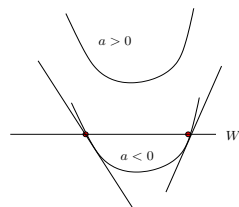
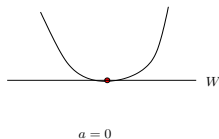
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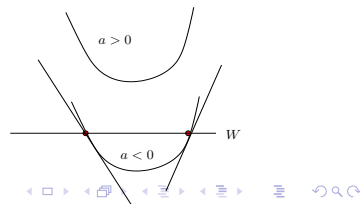
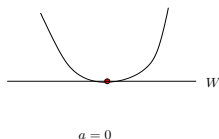
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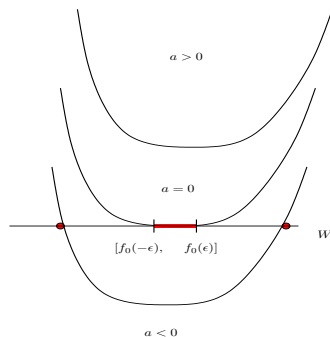
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Openness of Transversal Intersection

Theorem

Let \mathcal{A}, X, Y be \mathcal{C}^1 manifolds with X finite dimensional, $W \subset Y$ a closed \mathcal{C}^1 submanifold, $K \subset X$ a compact subset of X , and $\rho_a : X \rightarrow Y$ be a family of maps such that the correspondence $ev(a, x) = \rho_a(x)$ is \mathcal{C}^r . Then the subset $\mathcal{A}_{KW} \subset \mathcal{A}$ defined by $\mathcal{A}_{KW} = \{a \in \mathcal{A} \mid \rho_a \pitchfork_x W \text{ for } x \in K\}$ is open.



Example for Openness of Transversal Intersection

In our example, $K = X = [-1, 1]$ is compact and $\mathcal{A}_{KW} = \mathcal{A}_W := \{a \neq 0\}$. The set \mathcal{A}_{KW} is open in $\mathcal{A} = \mathbb{R}$ if a small enough neighborhood, $N_\epsilon(a)$, of the value $a \in \mathcal{A}_{KW}$, has $N_\epsilon(a) \subset \mathcal{A}$. That is, for any $a \neq 0$, there is a small interval $(a - \epsilon, a + \epsilon) \subset \mathcal{A}$.



Theorem

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Let $G = M \circ F$ be a GA map. For a typical mixing operator $M \in \mathcal{M}$, G has finitely many fixed points.



Set Up

- $\mathcal{A} = \mathcal{M}$, where \mathcal{M} is the set of mixing operators.
- $X = \Lambda \subset \mathbb{R}^n$ and $Y = \mathbb{R}_0^n := \{x \in \mathbb{R}^n \mid \sum_i x_i = 0\}$.
- Let $M \in \mathcal{M}$ and $x \in \Lambda$.

- Define $\rho_M(x) : \Lambda \rightarrow \mathbb{R}_0^n$, by

$$\rho_M(x) := (M \circ F - I)x.$$

- Define $\text{ev}_\rho(M, x) : \mathcal{M} \times \Lambda \rightarrow \mathbb{R}_0^n$ by

$$\text{ev}_\rho(M, x) := \rho_M(x) = M(F(x)) - x.$$

- $F, M, G, \rho_M, \text{ev}_\rho$ are each \mathcal{C}^1 .



Lemma

Lemma

For $\text{ev}_\rho(M, x) := M(F(x)) - x$, $\text{rank}(d(\text{ev}_\rho)) = n - 1$.



Proof

Since $\Lambda \subset \mathbb{R}^{n+}$,

$$d \operatorname{ev}_\rho = D \operatorname{ev}_\rho.$$

Similarly,

$$D(\operatorname{ev}_\rho|_{(\mathcal{M} \times \Lambda)}) = D \operatorname{ev}_\rho|_{T_{(P,y)}(\mathcal{M} \times \Lambda)}.$$

Because $T(\mathbb{R}_0^n) = \mathbb{R}_0^n$, and

$$T(\mathcal{M} \times \Lambda) = \{(P, y) \mid P = (P_1, \dots, P_n) \text{ with } \sum_i P_i = 0 \text{ and } y \in \mathbb{R}^n\},$$

it suffices to show $D \operatorname{ev}_\rho : T(\mathcal{M} \times \Lambda) \rightarrow \mathbb{R}_0^n$ is onto. By computation, for any $z \in \mathbb{R}_0^n$, and $(M, x) \in \mathcal{M} \times \Lambda$, there exists $(P, y) \in T(\mathcal{M} \times \Lambda)$ so that

$$D \operatorname{ev}_{(M,x)}(P, y) = z.$$



Lemma

Lemma

Let $\text{ev}_\rho : \mathcal{M} \times \Lambda \rightarrow \mathbb{R}^n$, $x \in \Lambda$, and $M \in \mathcal{M}$. Then $\text{ev}_\rho \pitchfork \{0\}$.

Proof: Show Definition of Transversality.

$W := \{0\}$, so $T_0W = \{0\}$. Need $(T_x \text{ev}_\rho)(T_x X) = T_0Y = \mathbb{R}^{n-1}$ to show

$$(T_x \text{ev}_\rho)(T_x X) + T_0W = T_0Y$$

$$\mathbb{R}^{n-1} + 0 = \mathbb{R}^{n-1}.$$

So, we show $D\text{ev}_\rho(x)$ is surjective. By the above Lemma, $\text{rank}(D\text{ev}_\rho(x)) = n - 1$, and therefore $D\text{ev}_\rho(x)$ is surjective and $\text{ev}_\rho \pitchfork \{0\}$. □



Proposition

Proposition

Let $\mathcal{M}_{\{0\}} := \{M \in \mathcal{M} \mid \rho_M \pitchfork \{0\}\}$. Then $\mathcal{M}_{\{0\}}$ is dense in \mathcal{M} . That is, the set of parameter values for which ρ_M is transversal to $\{0\}$ is dense in \mathcal{M} .

Proof: Use Transversal Density Theorem...

- 1 By above Lemma, $\text{ev}_\rho \pitchfork \{0\}$.
- 2 $\dim(X = \Lambda) = m < \infty$ and $\text{codim}(W = \{0\})$ in $Y = \mathbb{R}_0^n$ is $q = n - 1 < \infty$.
- 3 $r > \max(0, m - q)$. Since $r = 1$, clearly $r > \max(0, 0) = 0$.





Proposition

Proposition

The set $\mathcal{M}_{\{0\}}$ is open in \mathcal{M} .

Proof: Openness of Transversal Intersection Theorem.

\mathcal{M}, X, Y are \mathcal{C}^1 manifolds and the maps $\rho_M(x)$ are \mathcal{C}^1 . We take $K = X = \Lambda$, and thus K is a compact subset of the finite dimensional manifold X . Similarly $W = \{0\} \subset Y$ is closed. Thus,

$$\mathcal{M}_{K\{0\}} = \{M \in \mathcal{M} \mid \rho_M \pitchfork_x \{0\} \text{ for } x \in K = X\}$$

is open in \mathcal{M} . □





Proposition

Proposition

For generic $M \in \mathcal{M}$,

- 1 $\rho_M \pitchfork \{0\}$. That is, the set of parameter values for which ρ_M is transversal to $\{0\}$ is open and dense in \mathcal{M} .
- 2 The set of parameter values for which $\rho_M^{-1}(\{0\})$ has finitely many solutions is open and dense in \mathcal{M} .



Proof of Proposition

Proposition

(1) For generic $M \in \mathcal{M}$, $\rho_M \pitchfork \{0\}$. That is, the set of parameter values for which ρ_M is transversal to $\{0\}$ is open and dense in \mathcal{M} .

Proof.

Result follows directly from previous two Lemmas. □



Proof of Proposition

Proposition

(2) *The set of parameter values for which $\rho_M^{-1}(\{0\})$ has finitely many solutions is open and dense in \mathcal{M} .*

Proof Idea: Use Corollary to the Transversal Density Theorem.

- By Corollary, $\rho_M^{-1}(\{0\})$ has finitely many connected components for an open and dense set of values $M \in \mathcal{M}$.
- We show each connected component contains only isolated points, and therefore each connected component is an isolated point.
- Finitely many connected components implies finitely many isolated points, and $\rho_M^{-1}(\{0\})$ has finitely many solutions.





Proof of Main Result

Theorem

Let $G = M \circ F$ be a GA map. For a typical mixing operator $M \in \mathcal{M}$, G has finitely many fixed points.

Proof.

By previous Lemma, for $M \in \mathcal{M}_{\{0\}} \subset \mathcal{M}$, $\rho_M(x) = 0$ has finitely many solutions in Λ . That is, for generic $M \in \mathcal{M}$,

$$\rho_M(x) = M(F(x)) - x$$

has finitely many solutions in Λ . Thus solutions to $\rho_M(x) = 0$ correspond to fixed points of $G = M \circ F$. □

